Linear systems – Final exam

Final exam 2024–2025, Tuesday 17 June 2025, 15:00 – 17:00

Instructions

1. The use of books, lecture notes, or (your own) notes is not allowed.
2. All answers need to be accompanied with an explanation or calculation.

Problem 1 (8 + 4 + 6 + 12 + 4 = 34 points)

Consider the model

$$
\begin{aligned}
A_1\dot{h}_1(t) &= -a_1\sqrt{2gh_1(t)} + u(t), \\
A_2\dot{h}_2(t) &= -a_2\sqrt{2gh_2(t)} + a_1\sqrt{2gh_1(t)},
\end{aligned}
\tag{1}
$$

describing the water height in two interconnected tanks. Particularly, for tank $i \in \{1, 2\}$, $h_i(t) \in \mathbb{R}$ is the water height, $A_i > 0$ its cross-sectional area, and $a_i > 0$ the cross-sectional area of the outlet. The input $u(t) \in \mathbb{R}$ represents the inflow to the first tank, and g denotes the graviational constant.

(a) Consider initial conditions $h_i(0) = h_{i,0}$ and assume that $h_{i,0} \geq 0$ for $i \in \{1, 2\}$. Moreover, assume that the inflow is constant and nonnegative, i.e., $u(t) = \bar{u}$ for all $t \geq 0$, with $\bar{u} \geq 0$. On the basis of the dynamics (1), explain that $h_1(t) \geq 0$ and $h_2(t) \geq 0$ for all $t \geq 0$.

(b) In the notation of (a), take $h_{i,0} > 0$ for $i \in \{1, 2\}$ and $\bar{u} = 0$. The total water volume in the two tanks reads

$$V(h_1, h_2) = A_1h_1 + A_2h_2.$$

Show that the total water volume cannot increase as a function of time.

(c) Take $u(t) = \bar{u}$ for all t, with $\bar{u} > 0$. Find an equilibrium point $(\bar{h}_1, \bar{h}_2)$ and show that this equilibrium point is unique.

(d) Linearize the system (1) around the equilibrium point

$$\left(\begin{bmatrix} \bar{h}_1 \\ \bar{h}_2 \end{bmatrix}, \bar{u} \right).$$

Hint. In the linearized dynamics, simply write $\bar{h}_i$ rather than the expressions obtained in (c).

(e) Is the linearized system obtained in (d) asymptotically stable?

Problem 2 (4 + 10 + 10 + 4 = 28 points)

Consider the linear system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t)\end{aligned} \qquad \text{with} \qquad A = \begin{bmatrix} -5 & 17 \\ -1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 3 \end{bmatrix},$$

and where $x(t) \in \mathbb{R}^2$, $u(t) \in \mathbb{R}$, and $y(t) \in \mathbb{R}$.

(a) Verify that the system is observable.

(b) Find a nonsingular matrix T and real numbers α_1, α_2 such that

$$TAT^{-1} = \begin{bmatrix} 0 & \alpha_1 \\ 1 & \alpha_2 \end{bmatrix}, \quad CT^{-1} = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

(c) Use the matrix T from (b) to design a stable state observer of the form

$$\dot{\xi}(t) = A\xi(t) + Bu(t) + G(y(t) - C\xi(t)).$$

In particular, ensure that $\sigma(A - GC) = \{-2, -3\}$.

(d) Can the system be stabilized by dynamic output feedback?

Problem 3 (12 points)

Consider matrices $A \in \mathbb{R}^{n \times n}$ and $C \in \mathbb{R}^{p \times n}$. Let $\lambda \in \mathbb{C}$ be an eigenvalue of A such that λ is not (A, C)-observable. Show that λ is an eigenvalue of $A - GC$ for any $G \in \mathbb{R}^{n \times p}$.

Problem 4 (16 points)

Consider the two linear systems

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad \dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}u(t),$$

with $x(t), \bar{x}(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$. Assume that both systems are controllable and, in addition, that they are similar, i.e., there exists a nonsingular $T \in \mathbb{R}^{n \times n}$ such that

$$\bar{A} = TAT^{-1}, \qquad \bar{B} = TB.$$

Show that T is unique.

(10 points free)